MANIPAL UNIVERSITY

Deemed- to -be -University under Section 3 of the UGC Act, 1956

Reg. No.									

DEPARTMENT OF SCIENCES, M.Sc (P/C/M/G)
II SEMESTER END SEMESTER EXAMINATIONS, APRIL 2017

SUBJECT: **LINEAR ALGEBRA** [MAT 606]

(REVISED CREDIT SYSTEM)

Time: 3 Hours   Date: 25.04.2017   MAX. MARKS: 50

Note: (i) **Answer all FIVE full questions**

(ii) **All questions carry equal marks (3+3+4).**

1. A) Show that the following are equivalent for an $(n \times n)$ matrix A.

(i) A is invertible

(ii) The homogeneous system of equation $AX = 0$ has only the trivial solution $X = 0$.

(iii) The system of equations $AX = Y$ has a solution X for each $(n \times 1)$ matrix Y.

B) Let Vbe a finite dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and $\beta_1, \beta_2, \dots, \beta_n$ be any vectors in W. Then show that there is precisely one linear transformation T from V to W such that $T(\alpha_i) = \beta_i$, $i = 1,2, .. n$.

C) Let W_1 and W_2 are finite dimensional subspaces of a vector space $V.T$hen show that $(W_1 + W_2)$ is also finite dimensional and

$dim\, W_1 + dim\, W_2 = \dim(W_1 + W_2) + dim(W_1 \cap W_2)$.

2. A) Let T be a linear operator on a finite dimensional vector space V. Let $C_1, C_2, \dots, C_k$ be the distinct characteristic values of T and W_i be the null space of $(T - C_i I)$. Then show that the following are equivalent

(i) T is diagonalizable

(ii) The characteristic polynomial for T is

$f = (x - C_1)^{d_1}(x - C_2)^{d_2} \dots (x - C_k)^{d_k}$ and $\dim W_i = d_{i,} i = 1,2, \dots k$.

(iii) $\dim W_1 + \dim W_2 + \dots + \dim W_k = \dim V$.

B) Define an affine combination. Show that a subset A of a vector space V is a flat if and only if A is closed under affine combinations. .

C) Let $n > 1$ and let D be an alternating $(n-1)$-linear function on $(n-1) \times (n-1)$ matrices over K, a commutative rings with identity. Then show that, for each $j, 1 \leq j \leq n,$ the function $E_j$is defined by$E_j(A) = \sum_{i=1}^{n}(-1)^{i+j}A_{ij}D_{ij}(A)$ is an alternating n-linear function on $n \times n$ matrices A. Also show that if D is a determinant function, so is each E_j.

3. A) Let e be an elementary row operation and E be the corresponding elementary matrix of order $m \times m$. Then show that, for every $m \times n$ matrix A, $e(A) = EA$..

B) Let A be an $m \times n$ matrix with entries in a field F. Then show that the $row\ rank(A) = column\ rank(A)$.

C) Show that the subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

4. A) Let S and T be subspaces of a vector space V over a field F. Then show that the sum S+T is direct if and only if there exists bases A of S and B of T such that AUB is a basis for $S + T$.

B) Let V and W be vector spaces over the field F, and let T be a linear transformation from V a intoW. Then, show that the null space of T^t is the annihilator of the range of T. Also show that if V a and W are finite-dimensional, then $Rank\ (T^t) = Rank(T)$.

C) Let V and W be n- dimensional and m- dimensional vector spacesover the same field F, respectively. Then show that $L(V, W)$ is finite dimensional and has dimension mn.

5. A) Let K be a commutative ring with identity, and let A and B be $n \times n$ matrices over K. Then show that $\det(AB) = \det(A)\,.\det(B)$..

B) Let V be an inner product space, then show that , for every vectors α, β in V and $c \in F$,

(i) $|< \alpha, \beta >| \leq \|\alpha\|\|\beta\|$ (ii) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

C) Consider the three linear transformations on R^4 given by,

$f(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 2x_3 + x_4$, $f(x_1, x_2, x_3, x_4) = 2x_2 + x_4$

$f(x_1, x_2, x_3, x_4) = -2x_1 - 4x_3 + 3x_4$. Find the subspace W annihilated by the above transformation.

6. A) Let R be the field of real numbers, and let D be a function on 2×2 matrices over R, with values in R, such that $D(AB) = D(A)D(B)$ for all A, B. Suppose also that $D\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \neq D\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$. Prove the following.

(a) $D(0) = 0$;
(b) $D(A) = 0$ if $A^2 = 0$;
(c) $D(B) = -D(A)$ if B is obtained by interchanging the rows of A;
(d) $D(A) = 0$ if one row of A is 0.

B) Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. Check whether A is diagonalizable by computing the dimensions of the eigen spaces and find its minimal polynomial of A.

C) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Then show that T is diagonalizable if and only if the minimal polynomial for T has the form $p(x) = (x - c_1)(x - c_2) \ldots (x - c_k)$ where $c_1, c_2, \ldots, c_k$ are distinct elements of F.

****