Reg. No. | | | | | | | | | | |

MANIPAL INSTITUTE OF TECHNOLOGY
MANIPAL

A Constituent Institution of Manipal University

SEMESTER I M.TECH (AERO/CS) END SEMESTER EXAMINATION, MAR 2021
SUBJECT: APPLIED LINEAR ALGEBRA AND PROBABILITY – MAT 5156
(05-03-2021)

Time: 3 Hours Max. Marks : 50

Answer all the questions.

1A. Check the consistency of the following system and solve using Gaussian elimination:

$$\begin{aligned} x - 2y + 4z &= 8 \\ x + 8z &= 8 \\ 3y + 5z &= -1 \\ 2x - 9y + z &= 19. \end{aligned}$$

1B. Let V be a vector space over a field F, and let B be a subset of V. Prove that B is a basis of V if and only if every vector in V can be represented as a linear combination of vectors in B in a unique way.

1C. Compute the singular value decomposition of the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

$(\mathbf{4+3+3})$

2A. Let

$$A = \begin{bmatrix} 3 & -3 & -1 \\ 1 & -1 & -1 \\ -2 & 2 & 0 \end{bmatrix}.$$

(i) Find all eigenvalues of A.

(ii) Compute the minimal polynomial of A and hence determine whether A is diagonalisable.

2B. Let $V = \mathbb{R}^2$ be the real 2-tuple space and $W = \mathbb{R}^3$ the real 3-tuple space. Let $T\colon V \to W$ be the linear transformation that maps the vector $(2,3) \in V$ to the vector $(-4,3,3) \in W$ and maps the vector $(1,2) \in V$ to the the vector $(-3,2,1) \in W$. Find the matrix of T with respect to the standard bases of V and W. What is the rank of T?

2C. Let E be an idempotent matrix (a square matrix such that $E^2 = E$).

(i) Show that $I - E$ is also idempotent.

(ii) If E is of order $n \times n$, prove that trace $A \leq n$.

$(\mathbf{4+4+2})$

3A. Using elementary row transformations, compute the rank and nullity of A and find a basis for the null space of A.

$$A = \begin{bmatrix} 1 & 0 & 3 & -1 & -2 \\ 3 & 1 & 6 & 0 & -3 \\ -2 & 2 & -8 & 2 & 6 \\ 0 & -1 & 5 & -6 & -5 \end{bmatrix}$$

3B. Let V be the vector space of all 2×1 vectors over the field of real numbers, and let $M = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$. For $u, v \in V$, define $\langle u, v\rangle = v^T M u$. Verify that $\langle\, ,\, \rangle$ is an inner product on V. Find a basis for the subspace of all vectors in V that are orthogonal to the vector $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ under this inner product.

$(\mathbf{5} + \mathbf{5})$

4A. Using Gram-Schmidt orthogonalisation, derive an orthonormal basis for $\mathbb{R}^3$ from the basis $\{(0, 1, 1), (1, 0, -2), (1, -2, 2)\}$.

4B. If $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, compute A^8 using Cayley-Hamilton theorem.

4C. Prove that if V is an inner product space, then any set of orthonormal vectors in V is linearly independent. Also give an example of a linearly independent set that is not orthogonal.

$(\mathbf{4} + \mathbf{3} + \mathbf{3})$

5A. Box A contains 3 tags marked A and 1 tag marked B. Box B contains 2 tags marked A and 3 tags marked B. One of the two boxes is selected at random, and a tag is drawn from it.

(a) Find the probability that labels on the box and the tag drawn from it match.

(b) Find the probability that the tag drawn is B.

(c) If it is given that the tag drawn is B, find the probability that it was drawn from Box B.

5B. A and B are two events in a sample space with $P(A) = \frac{3}{4}$ and $P(B) = \frac{3}{8}$. Show that

$$\frac{1}{8} \leq p(A \cap B) \leq \frac{3}{8}.$$

5C. A coin with probability $\frac{1}{3}$ for heads and $\frac{2}{3}$ for tails is tossed until a head is obtained. Find the probability that the number of tosses required is even.

$(\mathbf{6} + \mathbf{2} + \mathbf{2})$