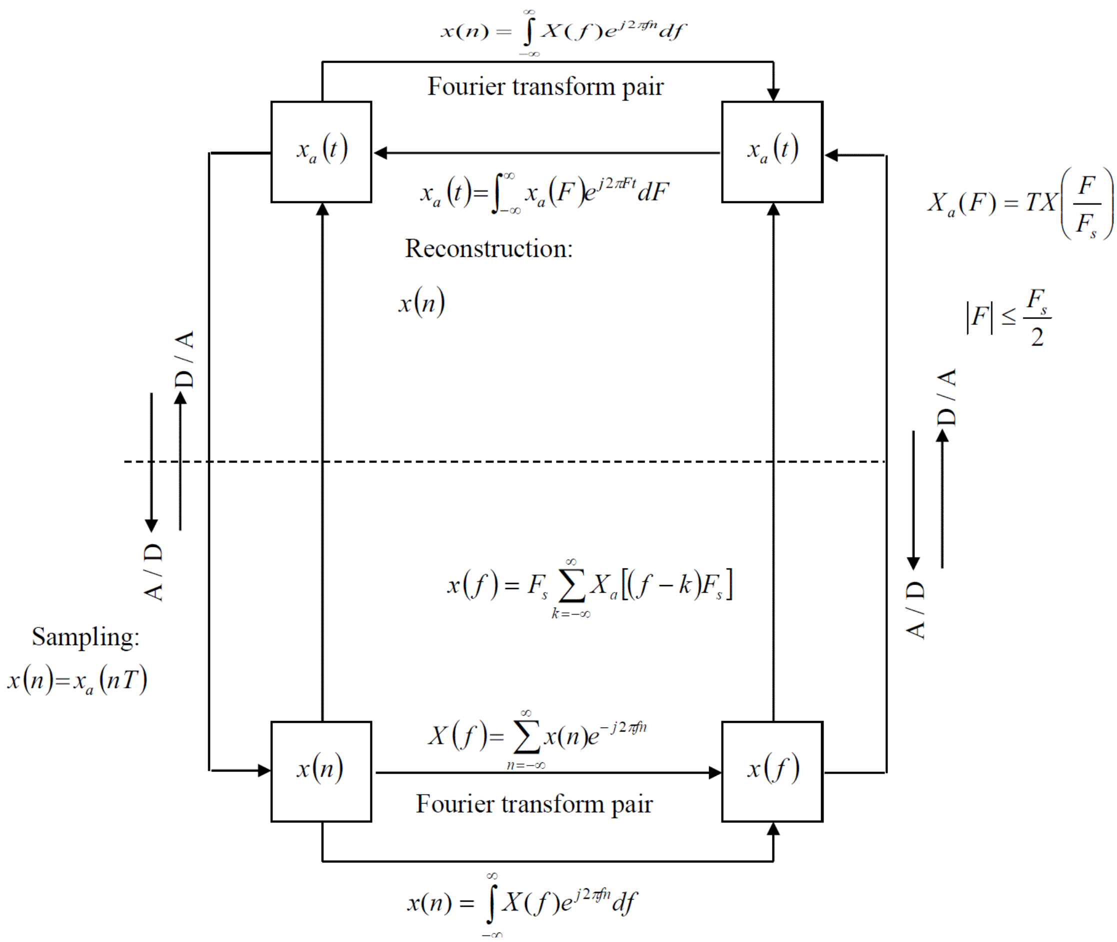
ECE 3151: Analog and Digital Communication

Time domain and frequency domain relationships for sampled signals



Fourier Series Representation of Continuous-Time Periodic Signals

Dirichlet Conditions:

1. x(t) is absolutely integrable over a period. i.e. $\int_{(T)} |x(t)| dt < \infty$

2. x(t) can have only finite number of maxima and minima over a period.

3. x(t) can have only a finite number of discontinuities in one period.

Definition: $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_{(T)} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\omega_0 = \frac{2\pi}{\mathrm{T}} \; ; \quad \mathrm{T} = \textit{fundamental period}$$

Properties of Continuous-Time Fourier Series

Property	Signal	FS Coefficient
	x(t), y(t)	a_k, b_k
1. Linearity	A*x*(t)+B*y*(t)	Aa_k+Bb_k
2. Time shifting	*x*(t-t_0)	$e^{-jk\omega_0 t_0}\ a_k$
3. Frequency Shifting	$e^{jM\omega_0 t}\ x(t)$	a_{k-M}
4. Time Reversal	*x*(-t)	a_{-k}
5. Time scaling	$x(\alpha\ t)$; $\alpha > 0$ period=T/α	a_k
6. Multiplication	*x*(t)y(t)	$c_k = \sum_{m=-\infty}^{\infty} a_m\, b_{k-m}$
7. Differentiation	$\frac{d}{dt}\,x(t)$	$jk\omega_0\ a_k$
8. Integration	$\int_{-\infty}^{t} x(t)\,dt$	$\frac{a_k}{jk\omega_0}$; a_0=0
9. Periodic Convolution	$\int_{(T)} x(\theta)y(t-\theta)d\theta$	$Ta_k b_k$
10. Conjugation	*x**(t)	a^*_{-k}

Parseval's Relation $$P = \frac{1}{T}\int_{(T)} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Fourier series (for periodic signals)

$$g_p(t) = a_0 + 2\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right]$$

$$\textit{where}\ \frac{n}{T_0}\ \textit{is the}\ n^{th}\ \textit{harmonic of fundamental frequency}, a_n\ \textit{and}\ b_n\ \textit{are amplitudes}, a_0\ \textit{is the DC value}$$
$$\textit{and}\ g_p(t)\ \textit{is the periodic signal.}$$

Complex Fourier series

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{j2\pi nt}{T_0}\right)$$

$$c_n = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} g_p(t)\exp\left(\frac{-j2\pi nt}{T_0}\right) \textit{where}\ c_n\ \textit{is the complex fourier co-efficient}.$$

Fourier Transform of Continuous-Time Signals

Definition:	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$

Signal	Fourier Transform
$\delta(t)$	1
$x(t) = 1$	$2\pi\delta(\omega)$
$sgn\,t$	$\begin{cases} \frac{2}{j\omega}; \omega \neq 0 \\ 0; \omega = 0 \end{cases}$
$u(t)$	$\frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)$
$e^{-at}u(t);\ a > 0$	$\frac{1}{\mathrm{a} + \mathrm{j}\omega}$
$e^{at}u(-t);\ a > 0$	$\frac{1}{a - j\omega}$
$e^{-a\|t\|}$	$\frac{2a}{a^2 + \omega^2}$
$x(t) = \begin{Bmatrix} 1; \|t\| \leq a \\ 0; \|t\| > a \end{Bmatrix}$	$2\left\{\frac{\sin \omega a}{\omega}\right\}$
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{Bmatrix} 1; \|\omega\| \leq W \\ 0; \|\omega\| > W \end{Bmatrix}$
$\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t};\ \omega_0 = \frac{2\pi}{T}$	$2\pi \sum_{\mathrm{k}=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$

Properties of Fourier Transform

Property	Signal $x(t)$, $y(t)$	Fourier Transform $X(j\omega)$, $Y(j\omega)$
1. Linearity	$Ax(t)+By(t)$	$AX(j\omega)+BY(j\omega)$
2. Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
3. Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4. Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(j\frac{\omega}{a})$
5. Time Reversal	$x(-t)$	$X(-j\omega)$
6. Differentiation in time	$\frac{d}{dt}x(t)$	$j\omega\, x(j\omega)$
7. Differentiation in frequency	$t\,x(t)$	$j\frac{d}{d\omega}X(j\omega)$
8. Integration in time	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)$
9. Duality	$X(t)$	$2\pi\, x(-\omega)$
10. Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
11. Modulation	$x(t)y(t)$	$\frac{1}{2\pi}\left(X(j\omega)*Y(j\omega)\right)$
12. Conjugation	$x^*(t)$	$X^*(-j\omega)$

Note: If integration is w.r.t frequency then

$$G(f)=\int_{-\infty}^{\infty}g(t)\exp(-j2\pi ft)dt\,, \quad g(t)=\int_{-\infty}^{\infty}G(f)\exp(j2\pi ft)df$$

Examples:

1) $\operatorname{sgn}(t)\Leftrightarrow\frac{1}{j\pi f}$

2) $e^{at}u(t)\Leftrightarrow\frac{1/a}{1+\frac{j2\pi f}{a}}$

3) $rect(t) \Leftrightarrow \sin c(f)$

4) $rect(\frac{t}{T}) \Leftrightarrow \sin c(fT)$

5) Autocorrelation of energy signal g(t)= $R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)d\tau$

6) $R_g(\tau) \Leftrightarrow \psi_g(f)$ *where* $\psi_g(f)$ is energy spectral density

7) Einsten-Weiner Khintchine relation(for power signals)

$$R_g(\tau) = \int_{-\infty}^{\infty} s_g(f)\exp(j2\pi f\tau)df \qquad s_g(f) = \int_{-\infty}^{\infty} R_g(\tau)\exp(-j2\pi f\tau)d\tau$$

Parseval's Relation

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Discrete-Time Fourier Series

Definition: $x[n] = \sum_{k=\langle N\rangle} X[k]e^{jk\Omega_0 n}$ $\qquad X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_0 n}$

$\Omega_0 = 2\pi/N$ $\qquad N = \textit{fundamental period}$

Parseval's Relation

$$P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |X[k]|^2$$

$$X(e^{j\Omega}) \circledast Y(e^{j\Omega}) = \int_{(2\pi)} X\left(e^{j\lambda}\right)Y\left(e^{j(\Omega-\lambda)}\right)d\lambda$$

Parseval's Relation

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{(2\pi)} \left|X(e^{j\Omega})\right|^2 d\Omega$$

Parseval's Formulae

Continuous-Time Signal

		Fourier Representation	Parseval's Formula
Periodic	Power Signal	Fourier Series	$P=\frac{1}{T}\int_{(T)} \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$
Aperiodic	Energy Signal	Fourier Transform	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$

Discrete – Time Signal

Periodic	Power signal	DTFS	$P = \frac{1}{N}\sum_{n=(N)} \lvert x[n]\rvert^2 = \sum_{k=(N)} \lvert X[k]\rvert^2$
Aperiodic	Energy signal	DTFT	$E = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{(2\pi)} \lvert X(e^{j\Omega})\rvert^2 d\Omega$

Amplitude Modulation $s(t) = A_c(1 + k_a m(t))\cos(2\pi f_c t)$

where, s(t) AM signal, m(t) corresponds to message signal and fc is the carrier frequency.

Double side band suppressed carrier (DSB-SC) $s(t) = A_c \cos(2\pi f_c t).m(t)$

Frequency Modulation $s(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_0^t m(t)dt)$

Bessel function The nth order Bessel function is denoted by $J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$

$1)As \;\; \beta \to \infty, \quad |J_n(\beta)| \to 0$

$$2)For \;\; fixed \;\beta,\; J_{-n}(\beta) = \begin{cases} J_n(\beta), & for\; n\; even \\ -J_n(\beta), & for\; n\, odd, \end{cases}$$

$3)For\, small\, values\, of \;\; \beta \le 0.3 radians$

$$J_0(\beta) \approx 1;\; J_1(\beta) \approx \frac{\beta}{2} \,\&\, J_n(\beta) \approx 0, n > 1$$

$$4) \sum_{n=-\infty}^{\infty} |J_n(\beta)|^2 = 1 \;\; for\; all\; \beta.$$

<u>Table of Bessel Functions</u>

β	J0(β)	J1(β)	J2(β)	J3(β)	J4(β)	J5(β)	J6(β)	J7(β)	J8(β)	J9(β)	J10(β)
0	1	0	0	0	0	0	0	0	0	0	0
0.1	0.9975	0.0499	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.9900	0.0995	0.0050	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3	0.9776	0.1483	0.0112	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.9604	0.1960	0.0197	0.0013	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5	0.9385	0.2423	0.0306	0.0026	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.6	0.9120	0.2867	0.0437	0.0044	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.7	0.8812	0.3290	0.0588	0.0069	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.8	0.8463	0.3688	0.0758	0.0102	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.9	0.8075	0.4059	0.0946	0.0144	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.7652	0.4401	0.1149	0.0196	0.0025	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
1.1	0.7196	0.4709	0.1366	0.0257	0.0036	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
1.2	0.6711	0.4983	0.1593	0.0329	0.0050	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000
1.3	0.6201	0.5220	0.1830	0.0411	0.0068	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000
1.4	0.5669	0.5419	0.2074	0.0505	0.0091	0.0013	0.0002	0.0000	0.0000	0.0000	0.0000
1.5	0.5118	0.5579	0.2321	0.0610	0.0118	0.0018	0.0002	0.0000	0.0000	0.0000	0.0000
1.6	0.4554	0.5699	0.2570	0.0725	0.0150	0.0025	0.0003	0.0000	0.0000	0.0000	0.0000
1.7	0.3980	0.5778	0.2817	0.0851	0.0188	0.0033	0.0005	0.0001	0.0000	0.0000	0.0000
1.8	0.3400	0.5815	0.3061	0.0988	0.0232	0.0043	0.0007	0.0001	0.0000	0.0000	0.0000
1.9	0.2818	0.5812	0.3299	0.1134	0.0283	0.0055	0.0009	0.0001	0.0000	0.0000	0.0000
2	0.2239	0.5767	0.3528	0.1289	0.0340	0.0070	0.0012	0.0002	0.0000	0.0000	0.0000
2.1	0.1666	0.5683	0.3746	0.1453	0.0405	0.0088	0.0016	0.0002	0.0000	0.0000	0.0000
2.2	0.1104	0.5560	0.3951	0.1623	0.0476	0.0109	0.0021	0.0003	0.0000	0.0000	0.0000
2.3	0.0555	0.5399	0.4139	0.1800	0.0556	0.0134	0.0027	0.0004	0.0001	0.0000	0.0000
2.4	0.0025	0.5202	0.4310	0.1981	0.0643	0.0162	0.0034	0.0006	0.0001	0.0000	0.0000
2.5	-0.0484	0.4971	0.4461	0.2166	0.0738	0.0195	0.0042	0.0008	0.0001	0.0000	0.0000
2.6	-0.0968	0.4708	0.4590	0.2353	0.0840	0.0232	0.0052	0.0010	0.0002	0.0000	0.0000
2.7	-0.1424	0.4416	0.4696	0.2540	0.0950	0.0274	0.0065	0.0013	0.0002	0.0000	0.0000
2.8	-0.1850	0.4097	0.4777	0.2727	0.1067	0.0321	0.0079	0.0016	0.0003	0.0000	0.0000
2.9	-0.2243	0.3754	0.4832	0.2911	0.1190	0.0373	0.0095	0.0020	0.0004	0.0001	0.0000
3	-0.2601	0.3391	0.4861	0.3091	0.1320	0.0430	0.0114	0.0025	0.0005	0.0001	0.0000
3.1	-0.2921	0.3009	0.4862	0.3264	0.1456	0.0493	0.0136	0.0031	0.0006	0.0001	0.0000
3.2	-0.3202	0.2613	0.4835	0.3431	0.1597	0.0562	0.0160	0.0038	0.0008	0.0001	0.0000
3.3	-0.3443	0.2207	0.4780	0.3588	0.1743	0.0637	0.0188	0.0047	0.0010	0.0002	0.0000
3.4	-0.3643	0.1792	0.4697	0.3734	0.1892	0.0718	0.0219	0.0056	0.0012	0.0002	0.0000

3.5	-0.3801	0.1374	0.4586	0.3868	0.2044	0.0804	0.0254	0.0067	0.0015	0.0003	0.0001
3.6	-0.3918	0.0955	0.4448	0.3988	0.2198	0.0897	0.0293	0.0080	0.0019	0.0004	0.0001
3.7	-0.3992	0.0538	0.4283	0.4092	0.2353	0.0995	0.0336	0.0095	0.0023	0.0005	0.0001
3.8	-0.4026	0.0128	0.4093	0.4180	0.2507	0.1098	0.0383	0.0112	0.0028	0.0006	0.0001
3.9	-0.4018	-0.0272	0.3879	0.4250	0.2661	0.1207	0.0435	0.0130	0.0034	0.0008	0.0002
4	-0.3971	-0.0660	0.3641	0.4302	0.2811	0.1321	0.0491	0.0152	0.0040	0.0009	0.0002
4.1	-0.3887	-0.1033	0.3383	0.4333	0.2958	0.1439	0.0552	0.0176	0.0048	0.0011	0.0002
4.2	-0.3766	-0.1386	0.3105	0.4344	0.3100	0.1561	0.0617	0.0202	0.0057	0.0014	0.0003
4.3	-0.3610	-0.1719	0.2811	0.4333	0.3236	0.1687	0.0688	0.0232	0.0067	0.0017	0.0004
4.4	-0.3423	-0.2028	0.2501	0.4301	0.3365	0.1816	0.0763	0.0264	0.0078	0.0020	0.0005
4.5	-0.3205	-0.2311	0.2178	0.4247	0.3484	0.1947	0.0843	0.0300	0.0091	0.0024	0.0006
4.6	-0.2961	-0.2566	0.1846	0.4171	0.3594	0.2080	0.0927	0.0340	0.0106	0.0029	0.0007
4.7	-0.2693	-0.2791	0.1506	0.4072	0.3693	0.2214	0.1017	0.0382	0.0122	0.0034	0.0008
4.8	-0.2404	-0.2985	0.1161	0.3952	0.3780	0.2347	0.1111	0.0429	0.0141	0.0040	0.0010
4.9	-0.2097	-0.3147	0.0813	0.3811	0.3853	0.2480	0.1209	0.0479	0.0161	0.0047	0.0012
5	-0.1776	-0.3276	0.0466	0.3648	0.3912	0.2611	0.1310	0.0534	0.0184	0.0055	0.0015

Noise: Mean square value of thermal noise, $E\{V_{TN}^2\} = E_n^2 = 4kTB_nR$

E_n = RMS Noise voltage B_n = Noise bandwidth in Hz

R=Resistance of conductor K=Boltzmann's constant=$1.38\times10^{-23}\ J/k$

Available thermal noise power, $P_n = kTB_n$

Friss's Formula: $F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2}$

F=Overall noise factor F_1, F_2, F_3 = Noise factor G=power gain of amplifiers.

Noise Temperature: $T_e = T_{e1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2}$

Where T_{e1}, T_{e2}, T_{e3} are noise temperature of individual stages.

Noise factor of a Lossy network=$\frac{available\, S/N\ power\, ratio\, at\, the\, input}{available\, S/N\ power\, ratio\, at\, the\, output}$

Gram-Schmidt Orthogonalization Procedure

Let $\vec{u}$ and $\vec{v}$ be two vectors, the projection of the vector $\vec{v}$ on $\vec{u}$ is defined as follows:

$$Proj_{\vec{u}}\vec{v} = \frac{(\vec{v}.\vec{u})}{|\vec{u}|^2}\vec{u}.$$

Consider a set of real-valued energy signals $s_1(t)$, $s_2(t)$, ……., $s_M(t)$.

Energy of the signal $s_1(t)$, $E_1 = \int_0^T s_1(t)^2\,dt$, with basis function $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$.

The intermediate function is

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t) \quad i = 1,2...M$$

where, $s_{ij} = \int_0^T s_i(t)\phi_j(t)dt \quad j = 1,2...i-1$

In general, M-ary modulation scheme, $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, ……… $\phi_N(t)$ are number of basis functions. $\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}} \quad i = 1,2...N$

Pulse Code Modulation

No. of bits to represent each quantized level, $m = \log_2 L$

For, the given input random variables 'X' with zero mean, variance = σ^2_X, quantization error 'Q' with zero mean, variance = σ^2_Q

Output SNR, $(SNR)_0 = \sigma_X^2/ \sigma_Q^2 = \sigma_X^2/(\Delta^2/12)$

$(SNR)_{0,\,dB} = 1.8 + 6m$ where 'm' → Minimum no. of bits needed to represent each sample

Total no. of quantization levels, $L = 2\,|x_{max}| / \Delta = 2^m$

Differential PCM

$(SNR)_0 = (\sigma_x^2 / \sigma_E^2). (\sigma_E^2 / \sigma_Q^2)$

$= G_P. (SNR)_P$

σ_E^2 → Variance of the prediction error, $e(nT_s)$

$(SNR)_P \rightarrow$ Prediction error-to-quantization noise ratio

$G_P \rightarrow$ Prediction gain produced by differential quantization

Delta Modulation

To avoid slope overload distortion, $\Delta/T_S \geq \max |dx(t)/dt|$ (or) $A_m \leq \Delta/(2\pi f_m T_S)$

The output SNR, $(SNR)_0 = 3/(8\pi^2 f_m^2 T_S^3 W)$

Robust Quantization

1. μ-law companding:

$$\frac{c(|x|)}{xmax} = \frac{\ln(1+\mu\frac{|x|}{xmax})}{\ln(1+\mu)}, \quad 0 \leq \frac{|x|}{xmax} \leq 1$$

$x \rightarrow$ input levels $\mu \rightarrow$ decides the amount of compression

If 'μ' is increasing, the amount of compression increases.

For $\mu=0 \rightarrow$ No compression and it leads to Uniform Quantization

In practical, PCM telephone systems in U.S, Canada, Japan uses $\mu = 255$

2. A-law companding:

$$\frac{c(|x|)}{xmax} = \frac{A\frac{|x|}{xmax})}{1+\ln A}, \quad 0 \leq \frac{|x|}{xmax} < \frac{1}{A}$$

$$= \frac{1+\ln(A\frac{|x|}{xmax})}{1+\ln A}, \quad \frac{1}{A} \leq \frac{|x|}{xmax} \leq 1$$

For A = 1, it is uniform quantization

In practical, PCM telephone systems in Europe uses A = 87.56

Matched Filter Receiver

The impulse response, $h_{opt}(t) = \phi(T - t)$

The transfer function, $H_{opt}(f) = \phi^*(f) \exp(-j2\pi fT)$

Properties of Matched Filter:

1. Fourier transform of the filter output, $\phi_0(t) = |\phi(f)|^2 \exp(-j2\pi fT)$

2. Matched filter output, $\phi_0(t) = R_\phi(t - T)$
   $R_\phi(t - T) \rightarrow$ Autocorrelation funcion of the input $\phi(t)$

3. The output SNR, $(SNR)_{0,\max} = E/(N_0/2)$
   $E \rightarrow$ Signal energy $\quad N_0/2 \rightarrow$ PSD of white noise

Digital Modulation techniques (table of error functions can be found on page 18)

Coherent Binary PSK: The average probability error, $P_e = \frac{1}{2}\,\mathrm{erfc}[\sqrt{(E_b/N_0)}]$

Coherent Binary FSK: The average probability error, $P_e = \frac{1}{2}\,\mathrm{erfc}[\sqrt{(E_b/2N_0)}]$

Non-coherent Binary FSK: The average probability error, $P_e = \frac{1}{2}\exp[(-E_b/2N_0)]$

The average probability error for DPSK, $P_e = \frac{1}{2}\exp[(-E_b/N_0)]$

The average probability error for QPSK, $P_e = \mathrm{erfc}[\sqrt{(E_b/N_0)}]$

$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-z^2}dz$ $\;\mathrm{erf}(u) = 1 - erfc(u)$ for large positive u, $erfc(u) = \frac{e^{-u^2}}{u\sqrt{\pi}}$

Power Spectra of Discrete PAM Signals

The discrete amplitude-modulated pulse train is described as random process,

$X(t) = \sum_{k=-\infty}^{\infty} Ak\, v(t - kT)$ where $A_k \rightarrow$ discrete random variable

$v(t) \rightarrow$ basic pulse shape centered at '0' $\quad T \rightarrow$ Symbol duration

Power Spectral Density

For NRZ Unipolar format, $S_X(f) = (a^2T_b/4)\mathrm{sinc}^2(fT_b) + (a^2/4)\,\delta(f)$

$\delta(f) \rightarrow$ Dirac delta function at $f = 0$

$\mathrm{sinc}(fT_b) = 0$ at $f = \pm 1/T_b, \pm 1/2T_b, \ldots\ldots\ldots$

Complementary Error Function Table

x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)
0	1.000000	0.5	0.479500	1	0.157299	1.5	0.033895	2	0.004678	2.5	0.000407	3	0.00002209
0.01	0.988717	0.51	0.470756	1.01	0.153190	1.51	0.032723	2.01	0.004475	2.51	0.000386	3.01	0.00002074
0.02	0.977435	0.52	0.462101	1.02	0.149162	1.52	0.031587	2.02	0.004281	2.52	0.000365	3.02	0.00001947
0.03	0.966159	0.53	0.453536	1.03	0.145216	1.53	0.030484	2.03	0.004094	2.53	0.000346	3.03	0.00001827
0.04	0.954889	0.54	0.445061	1.04	0.141350	1.54	0.029414	2.04	0.003914	2.54	0.000328	3.04	0.00001714
0.05	0.943628	0.55	0.436677	1.05	0.137564	1.55	0.028377	2.05	0.003742	2.55	0.000311	3.05	0.00001608
0.06	0.932378	0.56	0.428384	1.06	0.133856	1.56	0.027372	2.06	0.003577	2.56	0.000294	3.06	0.00001508
0.07	0.921142	0.57	0.420184	1.07	0.130227	1.57	0.026397	2.07	0.003418	2.57	0.000278	3.07	0.00001414
0.08	0.909922	0.58	0.412077	1.08	0.126674	1.58	0.025453	2.08	0.003266	2.58	0.000264	3.08	0.00001326
0.09	0.898719	0.59	0.404064	1.09	0.123197	1.59	0.024538	2.09	0.003120	2.59	0.000249	3.09	0.00001243
0.1	0.887537	0.6	0.396144	1.1	0.119795	1.6	0.023652	2.1	0.002979	2.6	0.000236	3.1	0.00001165
0.11	0.876377	0.61	0.388319	1.11	0.116467	1.61	0.022793	2.11	0.002845	2.61	0.000223	3.11	0.00001092
0.12	0.865242	0.62	0.380589	1.12	0.113212	1.62	0.021962	2.12	0.002716	2.62	0.000211	3.12	0.00001023
0.13	0.854133	0.63	0.372954	1.13	0.110029	1.63	0.021157	2.13	0.002593	2.63	0.000200	3.13	0.00000958
0.14	0.843053	0.64	0.365414	1.14	0.106918	1.64	0.020378	2.14	0.002475	2.64	0.000189	3.14	0.00000897
0.15	0.832004	0.65	0.357971	1.15	0.103876	1.65	0.019624	2.15	0.002361	2.65	0.000178	3.15	0.00000840
0.16	0.820988	0.66	0.350623	1.16	0.100904	1.66	0.018895	2.16	0.002253	2.66	0.000169	3.16	0.00000786
0.17	0.810008	0.67	0.343372	1.17	0.098000	1.67	0.018190	2.17	0.002149	2.67	0.000159	3.17	0.00000736
0.18	0.799064	0.68	0.336218	1.18	0.095163	1.68	0.017507	2.18	0.002049	2.68	0.000151	3.18	0.00000689
0.19	0.788160	0.69	0.329160	1.19	0.092392	1.69	0.016847	2.19	0.001954	2.69	0.000142	3.19	0.00000644
0.2	0.777297	0.7	0.322199	1.2	0.089686	1.7	0.016210	2.2	0.001863	2.7	0.000134	3.2	0.00000603
0.21	0.766478	0.71	0.315335	1.21	0.087045	1.71	0.015593	2.21	0.001776	2.71	0.000127	3.21	0.00000564
0.22	0.755704	0.72	0.308567	1.22	0.084466	1.72	0.014997	2.22	0.001692	2.72	0.000120	3.22	0.00000527
0.23	0.744977	0.73	0.301896	1.23	0.081950	1.73	0.014422	2.23	0.001612	2.73	0.000113	3.23	0.00000493
0.24	0.734300	0.74	0.295322	1.24	0.079495	1.74	0.013865	2.24	0.001536	2.74	0.000107	3.24	0.00000460
0.25	0.723674	0.75	0.288845	1.25	0.077100	1.75	0.013328	2.25	0.001463	2.75	0.000101	3.25	0.00000430
0.26	0.713100	0.76	0.282463	1.26	0.074764	1.76	0.012810	2.26	0.001393	2.76	0.000095	3.26	0.00000402
0.27	0.702582	0.77	0.276179	1.27	0.072486	1.77	0.012309	2.27	0.001326	2.77	0.000090	3.27	0.00000376
0.28	0.692120	0.78	0.269990	1.28	0.070266	1.78	0.011826	2.28	0.001262	2.78	0.000084	3.28	0.00000351
0.29	0.681717	0.79	0.263897	1.29	0.068101	1.79	0.011359	2.29	0.001201	2.79	0.000080	3.29	0.00000328
0.3	0.671373	0.8	0.257899	1.3	0.065992	1.8	0.010909	2.3	0.001143	2.8	0.000075	3.3	0.00000306
0.31	0.661092	0.81	0.251997	1.31	0.063937	1.81	0.010475	2.31	0.001088	2.81	0.000071	3.31	0.00000285
0.32	0.650874	0.82	0.246189	1.32	0.061935	1.82	0.010057	2.32	0.001034	2.82	0.000067	3.32	0.00000266
0.33	0.640721	0.83	0.240476	1.33	0.059985	1.83	0.009653	2.33	0.000984	2.83	0.000063	3.33	0.00000249
0.34	0.630635	0.84	0.234857	1.34	0.058086	1.84	0.009264	2.34	0.000935	2.84	0.000059	3.34	0.00000232
0.35	0.620618	0.85	0.229332	1.35	0.056238	1.85	0.008889	2.35	0.000889	2.85	0.000056	3.35	0.00000216
0.36	0.610670	0.86	0.223900	1.36	0.054439	1.86	0.008528	2.36	0.000845	2.86	0.000052	3.36	0.00000202
0.37	0.600794	0.87	0.218560	1.37	0.052688	1.87	0.008179	2.37	0.000803	2.87	0.000049	3.37	0.00000188
0.38	0.590991	0.88	0.213313	1.38	0.050984	1.88	0.007844	2.38	0.000763	2.88	0.000046	3.38	0.00000175
0.39	0.581261	0.89	0.208157	1.39	0.049327	1.89	0.007521	2.39	0.000725	2.89	0.000044	3.39	0.00000163
0.4	0.571608	0.9	0.203092	1.4	0.047715	1.9	0.007210	2.4	0.000689	2.9	0.000041	3.4	0.00000152
0.41	0.562031	0.91	0.198117	1.41	0.046148	1.91	0.006910	2.41	0.000654	2.91	0.000039	3.41	0.00000142
0.42	0.552532	0.92	0.193232	1.42	0.044624	1.92	0.006622	2.42	0.000621	2.92	0.000036	3.42	0.00000132
0.43	0.543113	0.93	0.188437	1.43	0.043143	1.93	0.006344	2.43	0.000589	2.93	0.000034	3.43	0.00000123
0.44	0.533775	0.94	0.183729	1.44	0.041703	1.94	0.006077	2.44	0.000559	2.94	0.000032	3.44	0.00000115
0.45	0.524518	0.95	0.179109	1.45	0.040305	1.95	0.005821	2.45	0.000531	2.95	0.000030	3.45	0.00000107
0.46	0.515345	0.96	0.174576	1.46	0.038946	1.96	0.005574	2.46	0.000503	2.96	0.000028	3.46	0.00000099
0.47	0.506255	0.97	0.170130	1.47	0.037627	1.97	0.005336	2.47	0.000477	2.97	0.000027	3.47	0.00000092
0.48	0.497250	0.98	0.165769	1.48	0.036346	1.98	0.005108	2.48	0.000453	2.98	0.000025	3.48	0.00000086
0.49	0.488332	0.99	0.161492	1.49	0.035102	1.99	0.004889	2.49	0.000429	2.99	0.000024	3.49	0.00000080

For NRZ Polar format, $S_X(f) = a^2 T_b \, sinc^2(fT_b)$

For NRZ Bipolar format, $S_X(f) = a^2 T_b \, sinc^2(fT_b) sin^2(\pi f T_b)$

Manchester format, $S_X(f) = a^2 T_b \, sinc^2(fT_b/2) sin^2(\pi f T_b/2)$

Intersymbol Interference (ISI)

Nyquist criterion for distortion less baseband transmission in the absence of noise and for ISI = 0 is

$$\sum_{n=-\infty}^{\infty} P(f - nRb) = Tb$$

$P_\delta(f)$ is Fourier of an infinite periodic sequence of delta functions of period 'T_b'

Information Theory (all logarithms are to the base 2)

The amount of information contained in symbol 'S_k' is $I(S_k) = \log(1/p_k)$

'p_k' → probability of occurance of symbol 'S_k'

For 'N' message symbols, Entropy(H) = $\sum_{k=0}^{N} p_k \log(1/p_k)$ bits/symbol

For the given two random variable X, Y

The joint entropy,

$$H(X, Y) = \sum_{k=1}^{m} \sum_{j=1}^{n} p(x_k, y_j) \log \frac{1}{p(x_k, y_j)}$$

The marginal entropies are

$$H(X) = \sum_{k=1}^{m} \sum_{j=1}^{n} p(x_k, y_j) \log \frac{1}{p(x_k)}$$

$$\text{Similarly, } H(Y) = \sum_{j=1}^{n} \sum_{k=1}^{m} p(x_k, y_j) \log \frac{1}{p(y_j)}$$

Conditional entropies are

$$H(X \mid Y) = E\{H(X \mid y_j)\}_j$$
$$= \sum_{j=1}^{n} p(y_j)\, H(X \mid y_j)$$
$$= \sum_{j=1}^{n} p(y_j) \sum_{k=1}^{m} p(x_k \mid y_j) \log \frac{1}{p(x_k \mid y_j)}$$

Or $$H(X \mid Y) = \sum_{j=1}^{n} \sum_{k=1}^{m} p(x_k, y_j) \log \frac{1}{p(x_k \mid y_j)}$$

$$H(Y \mid X) = \sum_{k=1}^{m} \sum_{j=1}^{n} p(x_k, y_j) \log \frac{1}{p(y_j \mid x_k)}$$

The different entropies are related as

$$H(Y \mid X) = H(X, Y) - H(X)$$

$$H(X, Y) = H(X) + H(Y \mid X)$$

$$H(X, Y) = H(Y) + H(X \mid Y)$$

$$H(X) \geq H(X|Y)$$

$$H(Y) \geq H(Y|X)$$

The mutual information is given as

$$I(X;Y) = \sum_x \sum_y p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)}$$
$$\text{or: } \sum_x \sum_y p(x,y) \log_2 \frac{p(x|y)}{p(x)}$$
$$\text{or: } I(X;Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y)$$

For a discrete memoryless source, coding efficiency, $\eta = H(X)/ L_{avg}$

$H(X) \rightarrow$ entropy

$L_{avg} \rightarrow$ minimum average number of bits/symbol

Also, $L_{avg} \geq H(X)$

Channel capacity, $C = B\log_2(1 + P/N_0B)$ bits/sec

$B \rightarrow$ Channel Bandwidth in Hz

$N_0/2 \rightarrow$ AWGN PSD limited to 'B' in Hz

$P \rightarrow$ Average Transmitting Power