Question Paper

Exam Date & Time: 06-Feb-2021 (10:00 AM - 01:15 PM)

MANIPAL ACADEMY OF HIGHER EDUCATION

MANIPAL SCHOOL OF INFORMATION SCIENCES, MANIPAL

FIRST SEMESTER MASTER OF ENGINEERING - ME (MACHINE LEARNING) DEGREE EXAMINATION - FEBRUARY 2021

Applied Linear Algebra [MCL 603]

Marks: 100 **Duration: 180 mins.**

SATURDAY, FEBRUARY 6, 2021

Answer all the questions.

1) [TLO 1.1, 1.2, L5] *Word count and word count histogram vectors.* Suppose the n-vector w is the word count vector associated with a document and a dictionary of n words. For simplicity we will assume that all words in the document appear in the dictionary. (10)

(a) What does $\mathbf{1}^T w$ tell us about the document?

(b) What does $w_{282} = 0$ mean for the document and the dictionary?

(c) Let h be the n-vector that gives the histogram of the word counts, i.e., h_i is the fraction of the words in the document that are word i in the dictionary. Express h in terms of w using a dot product. (You can assume that the document contains at least one word.)

2) [TLO 1.1, 1.2, 1.3, L6] *Orthogonality.* Suppose the n-vectors a and b are such that $a + b$ is orthogonal to $a - b$. Which of the following statements must always hold? Incorrect answers will carry negative points. (10)

(a) $\|a\| = \|b\|$.

(b) $\mathrm{rms}(a) = \mathrm{rms}(b)$.

(c) $\mathrm{std}(a) = \mathrm{std}(b)$.

3) [TLO 1.6, L4] *Projection.* Calculate the vector projection of the vector $a = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ on to the direction of the vectors: (10)

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

4) [TLO 1.6, L6] *Order of vectors in the Gram-Schmidt algorithm.* Suppose $\{a_1, a_2\}$ is a set comprising two linearly independent n-vectors. When we run the Gram-Schmidt algorithm on this set starting with a_1, we obtain the orthonormal vectors q_1, q_2. (10)

Now suppose we run the Gram-Schmidt algorithm starting with a_2. Do we get the orthonormal vectors q_2, q_1 (i.e., the orthonormal vectors obtained before in reverse order)?

If you believe this is true, give a very brief explanation why. If you believe it is not true, give a simple counter-example.

5) (10)

[TLO 2.3, L5] Suppose A is an $m \times n$-matrix. Explain the output of the following operations by clearly stating the dimension of the output:

(a) Ae_j.

(b) $(e_i + e_j)^{\mathrm{T}} A^{\mathrm{T}}$.

(c) $A\mathbf{1}$.

(d) $e_i^{\mathrm{T}} A e_j$.

6) [TLO 2.4, L4] Consider the RREF of an augmented matrix: (10)

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

(a) Is the underlying system of equations consistent?

(b) Identify the free and pivot variables.

(c) If the system is consistent, express the solution as a set of vectors.

7) [TLO 2.3, L5] *Down-sampling a signal.* Consider an n-vector x whose components x_i represent the value of a signal at time stamp $i = 1, 2, \ldots, n$. Assuming n is even, we want to construct a $2\times$ down-sampled version of x denoted as the $(n/2)$-vector y by multiplying x by an appropriate matrix A such that $y = Ax$ where: (10)

$$y_i = x_{2i}, \text{ for } i = 1, 2, \ldots, n/2.$$

Using $n = 6$, write the elements of matrix A.

8) [TLO 2.4, 2.7, L6] *Network tomography.* A network consists of n links, labeled $1, \ldots, n$. A path through the network is a subset of the links. (The order of the links on a path does not matter here.) Each link has a (positive) delay, which is the time it takes to traverse it. We let d denote the n-vector that gives the link delays which is not a directly measurable quantity. The total travel time of a path is the sum of the delays of the links on the path. Suppose our goal is to estimate the link delays (i.e., the vector d), from a large number of (noisy) measurements of the travel times along different paths. This data is given to you as an $N \times n$-matrix P, where (10)

$$P_{ij} = \begin{cases} 1, & \text{if link } j \text{ is on path } i, \\ 0, & \text{otherwise,} \end{cases}$$

and an N-vector t whose entries are the (noisy) travel times along the N paths. You can assume that $N > n$.

(a) What does the ith column of P tell us about the network?

(b) Write the travel times predicted by the sum of the link delays as a matrix-vector product.

(c) Write the RMS deviation (using matrix/vector terms) between the predicted travel times from the previous step and the measured travel times. This is the quantity we want to minimize in order to estimate the link delays d.

9) (10)

[TLO 2.3, 2.4, L6] *Recursive averaging.* Suppose that $u_1, u_2, \ldots$ is a sequence of n-vectors. Let $x_1 = 0$, and for $t = 2, 3, \ldots$, let x_t be the average of $u_1, \ldots, u_{t-1}$, i.e., $x_t = (u_1 + \cdots + u_{t-1})/(t-1)$. Express this as a linear dynamical system with input, i.e.,

$$x_{t+1} = A_t x_t + B_t u_t, \text{ for } t = 1, 2, \ldots,$$

(with initial state $x_1 = 0$) by clearly showing the elements of the matrices A_t and B_t.

Remark. This can be used to compute the average of an extremely large collection of vectors, by accessing them one-by-one.

10) [TLO 2.3, 2.4, 2.7, L6] Consider the following model for opinion formation among n individuals, each of whom interact with a certain number of individuals in the group. The numerical value of the ith person's opinion is denoted as x_i. The value of x_i is influenced by the following: (10)

- The ith person's self opinion denoted as s_i
- The opinions of the remaining individuals x_j, where $j = 1, 2, \ldots, n$ and $j \neq i$.

Assuming that the ith person gives a weightage w_{ij} to the jth person's opinion, we can compute x_i as follows:

$$x_i = \frac{s_i + \sum_{j \neq i} w_{ij} x_j}{1 + \sum_{j \neq i} w_{ij}}, \quad i = 1, \ldots, n. \tag{1}$$

It is clear that the weightage that a person gives to his own opinion is taken to be 1 as seen in the denominator of Equation (1).

(a) Rewrite Equation (1) as $(A+I)x = s$, where I represents the identity matrix. What are the elements of the matrix A and vector s?

(b) For a given weight matrix W whose entries are w_{ij}, and a vector of self opinion values s, we can compute the mean opinion as:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

In order to calculate x, say using Python's sympy library, what is the augmented matrix for which you will compute the RREF?

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