ECE 3152 Linear Control Theory

MATHEMATICAL MODELS OF SYSTEMS

Electrical Network Transfer Functions

Table: Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = \frac{V(s)}{I(s)}$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls

Note: The following set of symbols and units is used: $v(t)$ - V (volts); $i(t)$ – A (amps); $q(t)$ – Q (coulombs); C – F (farads), R – Ω (ohms); L – h (henries)

Translational Mechanical System Transfer functions

Table 2.4: Force-velocity, force-displacement, and impedance translational relationships for springs, viscous damper, and mass.

Component	Force-velocity	Force-displacement	Impedance $Z_m(s) = \frac{F(s)}{X(s)}$
Spring	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t))$	K
Viscous damper	$f(t) = Bv(t)$	$f(t) = B\frac{dx(t)}{dt}$	Bs
Mass	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used: $f(t)$ - N (newtons); $x(t)$ – m (meters); $v(t)$ – m/s (meters/second); K – N/m (newton/meter), B – N-s/m (newton-seconds/meter); M – kg (kilograms=newton-second^2/meter)

Rotational Mechanical System Transfer functions

Table 2.5: Torque-angular velocity, torque angular displacement, and impedance rotational relationships for springs, viscous damper, and inertia.

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_m(s) = \frac{T(s)}{\theta(s)}$
Spring	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t))$	K
Viscous damper	$T(t) = B\omega(t)$	$T(t) = B\frac{d\theta(t)}{dt}$	Bs
Inertia	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used: $T(t)$ – N-m (newton-meters); $\theta(t)$ – rad (radians); $\omega(t)$ – rad/s (radians/second); K – N-m/rad (newton-meter/second), B – N-m-s/rad (newton-meter-second/radians); J – kg-m^2 (kilogram-$meters^2$=newton-meter-$second^2$/radian)

Transfer Functions for Systems with Gears

$\boldsymbol{T_1}(t)$- input torque

$\theta_1(t)$ – angle of rotation of input gear

r_1 – radius of input gear

N_1 – number of teeth on input gear

$\boldsymbol{T_2}(t)$- output torque

$\theta_2(t)$ – angle of rotation of output gear

r_2 – radius of output gear

N_2 – number of teeth on output gear

a) $\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$

b) $T_1\theta_1 = T_2\theta_2$

c) $\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$

Note: Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio

$$\left[\frac{Number\ of\ teeth\ of\ gear\ on\ destination\ shaft}{Number\ of\ teeth\ of\ gear\ on\ source\ shaft}\right]^2$$

TRANSIENT RESPONSE SPECIFICATIONS

Second order system

$$\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

System is un-damped when $\xi = 0$

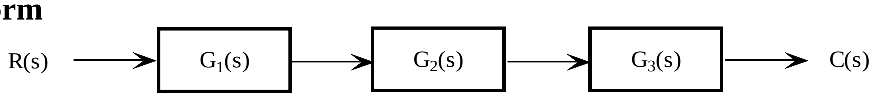
System is under damped when $0 < \xi < 1$

System is critically damped when $\xi = 1$

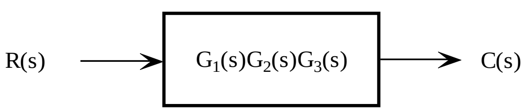
System is over damped when $\xi > 1$

REDUCTION OF MULTIPLE SUBSYSTEMS

Cascade form

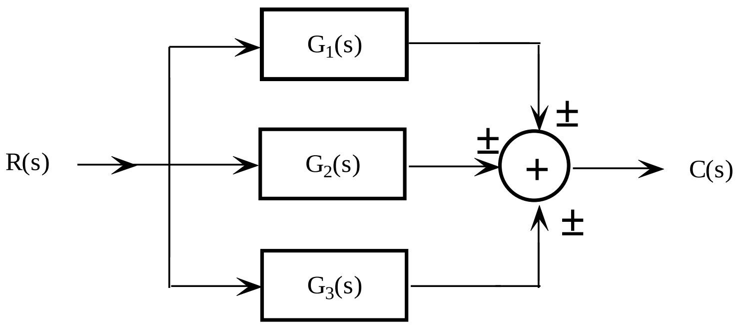


(a) Cascaded subsystem

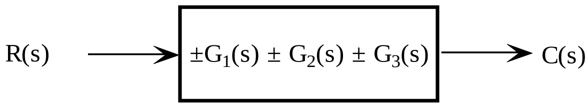


(b) Equivalent transfer function

Parallel form

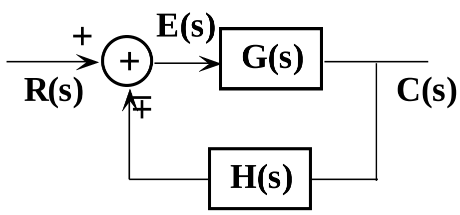


(a) Parallel subsystem

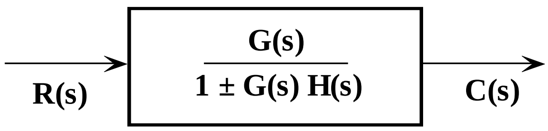


(b) Equivalent transfer function

Feedback Form



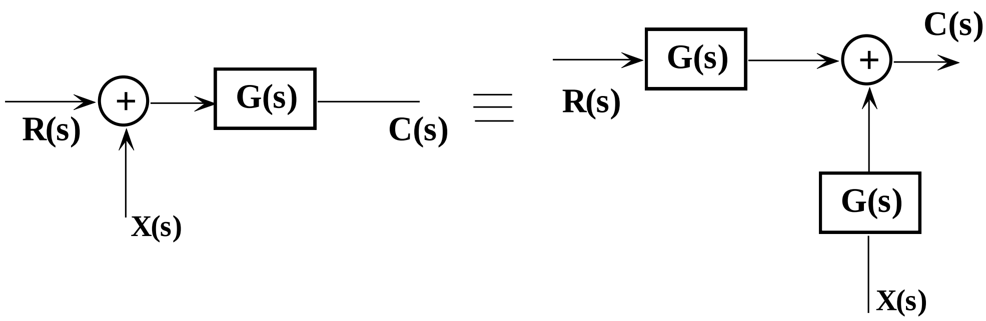
Feedback control system



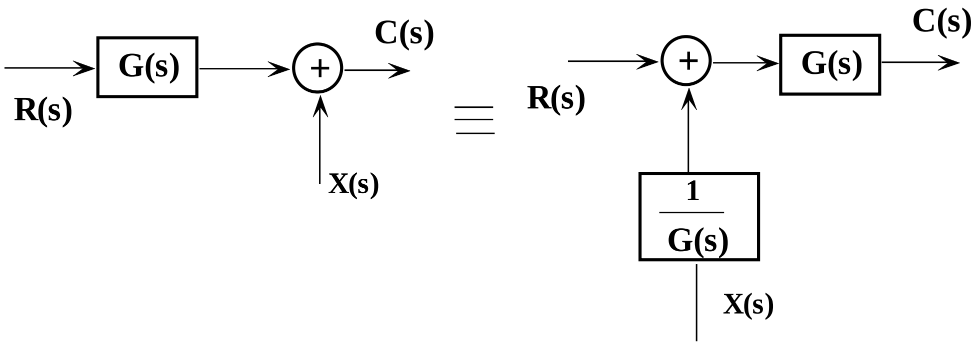
Equivalent transfer function

Moving blocks to create familiar forms

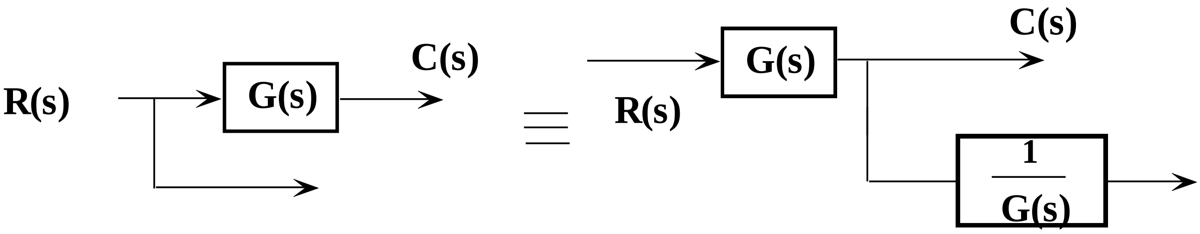
i) Moving a block to the left past a summing junction



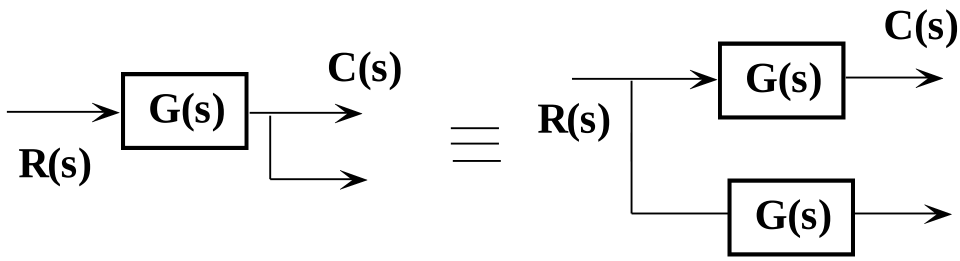
ii) Moving a block to the right past a summing junction



iii) **Moving a block to the left past a pickoff point**



iv) **Moving a block to the right past a pickoff point**



SIGNAL-FLOW GRAPHS
Mason's rule

The transfer function $\frac{C(s)}{R(s)}$ of a system represented by a signal-flow graph is

$$\frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

Where

k= number of forward paths

T_k= the kth forward-path gain

Δ = 1-∑ loop gains + ∑ product of non-touching loop gains taken two at a time - ∑ product of non-touching loop gains taken three at a time + ∑ product of non-touching loop gains taken four at a time …………….

Δ_k = Δ -∑ loop gain terms in Δ that touch the k_{th} forward path. Δ_k is formed by eliminating from Δ those loop gains that touch the k_{th} forward path.

STABILITY
Routh-Hurwitz criterion
Considering the characteristic equation
$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots \dots . + a_{n-1} s + a_n = 0$
Form the determinants from the coefficients of the characteristic equation. There are n determinants for n^{th} order system.
The system is stable if and only if the value of each determinant is greater than zero.

STEADY-STATE RESPONSE SPECIFICATIONS

Input	***Static error constants***	***Steady state errors***
Step function	$\boldsymbol{K_P = \lim_{s\to o} G(s)}$	$\frac{1}{1+k_p}$
Ramp function (t)	$\boldsymbol{K_r = \lim_{s\to o} S\, G(s)}$	$\frac{1}{k_v}$
Parabolic function (t^2)	$\boldsymbol{K_a = \lim_{s\to o} S^2 G(s)}$	$\frac{1}{k_a}$

FREQUENCY RESPONSE SPECIFICATIONS

a) Resonant peak $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$; $\xi = 0.707$
b) Resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$; $\xi < 0.707$
c) Bandwidth
$BW = \omega_n\sqrt{(1-2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}}$
d) Gain margin $= \frac{1}{|G(j\omega_p)H(j\omega_p)|}$;
Gain margin in $dB = 0 - gain\ in\ dB\ at\ \omega = \omega_p$
ω_p is phase crossover frequency
e) Phase margin=180^o + phase at $\omega = \omega_g$
ω_g is gain crossover frequency

ROOT LOCUS TECHNIQUES

a) Open-loop transfer function $G(s)H(s) = K\ \frac{N(s)}{D(s)}$
b) Magnitude criterion $|G(j\omega)H(j\omega)| = 1$
c) Angle criterion $\angle G(j\omega)H(j\omega) = \pm 180^o$
d) For breakaway and break-in points solve $\frac{dK}{ds} = 0$
e) Centroid of asymptote $\sigma = \frac{\sum open\ loop\ poles - \sum open\ loop\ zeros}{n-m}$
n= number of open loop poles; m= number of open loop zeros
f) Angle of intercept of asymptotes $\beta = \left(\frac{2l+1}{n-m}\right) 180^o; l = 0, 1, 2, \dots \dots . (n-m-1)$
g) Angle of departure from the complex pole $s = -a + jb$
$\emptyset = 180^o + Angle\ of\ \{(s + a - jb)G(s)H(s)\}$ at $s = -a + jb$
Angle of departure from the complex pole $s = -a - jb$
$\emptyset = 180^o + Angle\ of\ \{(s + a + jb)G(s)H(s)\}$ at $s = -a - jb$

h) Angle of arrival at the complex zero $s = -a + jb$

$\emptyset = 180^o - Angle\ of\ \left\{\frac{G(s)H(s)}{(s+a-jb)}\right\}$ at $s = -a + jb$

Angle of arrival at the complex zero $s = -a - jb$

$\emptyset = 180^o - Angle\ of\ \left\{\frac{G(s)H(s)}{(s+a+jb)}\right\}$ at $s = -a - jb$

i) Value of K at the point s=s_o on the root locus $K = \left|\frac{D(s_o)}{N(s_o)}\right|$

j) To find the jω axis crossing, use Routh Hurwitz criterion.

STATE-SPACE REPRESENTATION

a) State Equation: $\dot{X} = AX + Br$

b) Output equation: $y = CX + Dr$

c) Eigen Values: Solve $|\lambda I - A| = 0$

d) Transfer function: $T(s) = \frac{Y(s)}{R(s)} = C\,[sI - A]^{-1}\,B + D$

e) State transition matrix: $e^{At} = L^{-1}\,[sI - A]^{-1}$

f) Zero input response: $Y_{ZIR}(s) = C\,[sI - A]^{-1}\,X(0)$

g) Zero state response: $Y_{ZSR}(s) = C\,[sI - A]^{-1}\,B\,R(s)$

Gain margin = 20 log $\frac{1}{|\alpha|}$; where α is the magnitude of the function at $\varphi = -180^0$.

Phase Margin = ∠GH($j\omega_1$) +180^0; Where ω_1 is the gain cross over frequency.

Lag compensator $G_c(s)$ = Kc $\frac{s+1/T}{s+1/\alpha T}$; α >1

Lead compensator $G_c(s)$ =Kc $\frac{s+1/T}{s+1/\alpha T}$; α < 1

PID Controller $G_c(s)$ $= K_p + \frac{Ki}{s} + K_d s$

Laplace Transforms

Time domain	**Laplace Domain**
$\delta(t) =$ unit Impulse	1
u(t) = unit step	$\frac{1}{s}$
t = ramp	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
$e^{-at} - e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
sin(bt)	$\frac{b}{s^2+b^2}$
cos(bt)	$\frac{s}{s^2+b^2}$

$e^{-at}\sin(bt)$ ċ	$\dfrac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)$	$\dfrac{s+a}{(s+a)^2+b^2}$
Sinh(bt)	$\dfrac{b}{s^2-b^2}$
Cosh(bt)	$\dfrac{s}{s^2-b^2}$
$\delta(t-kT)$	e^{-kTs}
t^2	$\dfrac{2}{s^3}$
Initial value Theorem	$\lim_{\vec{t}\to 0} f(t) = \lim_{s\to\infty} sf(s)$
Final value Theorem	$\lim_{\vec{t}\to\infty} f(t) = \lim_{s\to 0} sf(s)$
First differentiation f`(t)	$sF(s) - f(0)$
Second differential f"(t)	$s^2F(s) - sf(0) - f^1(0)$