Question Paper

Exam Date & Time: 13-Jan-2023 (10:00 AM - 01:00 PM)

MANIPAL ACADEMY OF HIGHER EDUCATION

Manipal School of Information Sciences (MSIS), Manipal
First Semester Master of Engineering - ME (Artificial Intelligence & Machine Learning) Degree Examination - January 2023

Applied Linear Algebra [AML 5101]

Marks: 100 **Duration: 180 mins.**

Friday, January 13, 2023

Answer all the questions.

1) (10)

[10 points] [TLO 1.1, CO 1] Consider the vectors $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Calculate the following (you may leave the answer in terms of fractions, square root etc.):

(a) $\mathrm{rms}(x)$.

(b) $\mathrm{avg}(y)$.

(c) $\mathrm{std}(x)$.

(d) The correlation coefficient between x and y.

(e) The distance between x and y.

2) (10)

[10 points] [TLO 1.1, CO 1] The MAHE registrar has the complete list of courses taken by each graduating student in a program. This data is represented as a matrix X with m rows and n columns as follows:

Course / Student	1	2	$\cdots$	n
1	1	1	$\cdots$	0
2	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

m	1	0	$\cdots$	1

The entries of the data matrix are 1s and 0s representing whether a particular student has taken a particular course. For example, the red-highlighted entry 1 means that the 1st student has taken the 1st course and the blue-highlighted entry 0 means that the mth student has not taken the 2nd course. Recall that the ith student vector is represented as $x^{(i)}$ and the jth course vector is represented as x_j.

(a) The total number of courses the 2nd student has taken is $\boxed{?}^{\mathrm{T}}\boxed{?}$.

(b) In English, explain what the quantity $x_2 \cdot 1$ represents w.r.t. the data?

(c) The total number of students who have taken both class 5 and class 6 is $\boxed{?}^{\mathrm{T}}\boxed{?}$.

(d) In English, explain what the quantity $\left\|x^{(5)} - x^{(6)}\right\|^2$ represents w.r.t. the data?

(e) In English, explain what the quantity $\left[X^{\mathrm{T}}1\right]_3$ represents w.r.t. the data?

3) (10)

[TLO 1.2 CO 2] Suppose that $z_1, z_2, \ldots$ is a time series, with the number z_t giving the value in time t. For example, z_t could be the total sales at a particular store on day t. Consider the following model to predict the future sales z_{t+1} from the previous M sales values:

$$\hat{z}_{t+1} \approx \begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-(M-1)} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix},$$

where $\hat{z}_{t+1}$ denotes the model's prediction of z_{t+1} and the M-vector β contains the model coefficients which have to be computed. For this problem we will assume that $M = 10$. Thus, the model predicts tomorrow's value, given the values over the last 10 days. For each of the following cases of the model coefficients vector β, give a short interpretation in plain English as to what the model predicts as the future sales without referring to mathematical concepts like vectors, dot product, and so on:

(a) $\beta = e_1$;

(b) $\beta = 2e_1 - e_2$;

(c) $\beta = e_6$;

(d) $\beta = 0.5e_1 + 0.5e_2$.

4) (10)

[TLO 1.2, CO 2] Calculate the vector projection of the vector $a = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ on to the direction of the vectors:

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

5) (10)

[TLO 2.1 CO 2] Consider the 5×5-matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) For a 5-vector x, how are Ax and x related?

(b) Compute A^2 defined as the matrix-matrix product $A \times A$. Use the result to identify what A^5 would be without any further calculations.

6) (10)

[TLO 2.1, CO 2] Suppose A is an $m \times n$-matrix. For each operation below, give the output dimension and interpret the output in English in terms of the rows/columns/elements of A:

(a) $A(e_i - e_j)$.

(b) $e_i^{\mathrm{T}} A$.

(c) $A^{\mathrm{T}} \mathbf{1}$.

(d) $e_i^{\mathrm{T}} A^{\mathrm{T}} e_j$.

7) (10)

[TLO 2.1, CO 2] Consider an n-vector x whose components x_i represent the value of a signal at time stamp $i = 1, 2, \ldots, n$. We want to construct a $2\times$ up-sampled version of the signal x denoted as the $(2n-1)$-vector y using linear interpolation by multiplying x by an appropriate matrix A such that $y = Ax$, where:

$$y_i = \begin{cases} x_{\frac{i+1}{2}}, & \text{if } i \text{ odd}, \end{cases}$$

$$\left\lfloor \frac{1}{2}\left(x_{\frac{i}{2}} + x_{\frac{i}{2}+1}\right), \quad \text{if } i \text{ even.}\right.$$

Using $n = 5$, write the elements of matrix A.

8) (10)

[TLO 2.2, CO 2] Consider the RREF of an augmented matrix:

$$\begin{bmatrix} 1 & 0 & -7/3 & | & 1/3 \\ 0 & 1 & 8/3 & | & 1/3 \end{bmatrix}.$$

(a) Is the underlying system of equations consistent?

(b) Identify the free and pivot variables.

(c) If the system is consistent, express the solution as a set of vectors.

9) (10)

[TLO 3.1, CO 3] A train network consists of n stations and m paths, Each station has a delay associated with it which is the time it takes to traverse the station. The delay at each station is not a directly measurable quantity which we denote by the (unknown) n-vector d. However, we can measure the actual total travel time across any route which can be thought of as an approximate measure of the sum of the delays at all stations on that route. For example, if the 1st route contains stations 3, 7, 8, and 9, then the travel time across the 1st route is given by $\underbrace{t_1}_{\text{actual measured travel time}} \approx \underbrace{d_3 + d_7 + d_8 + d_9}_{\text{predicted travel time}}$.

Suppose our goal is to estimate the unknown station delays (that is, the vector d), from a large number of (noisy) measurements of the travel times along all the routes. The associated data is given to you as follows:

- an $m \times n$-matrix P, where

$$P_{ij} = \begin{cases} 1, & \text{if station } j \text{ is on route } i, \\ 0, & \text{otherwise;} \end{cases}$$

- an m-vector t whose entries are the (noisy) measured travel times along the m routes.

You can assume that $m > n$; that is, there are more routes than stations.

(a) What does the jth column of P tell us about the train network?

(b) What does the quantity $P1$ tell us about the train network?

(c) Write the travel times predicted for all routes as a matrix-vector product.

(d) In order to estimate the unknown station delays d, we want to minimize the deviation between the predicted and the actual travel times. One way to do that is minimize the

$$\underbrace{\text{mean/rms/std/norm/max/min}}_{\text{choose the correct option}} \text{ of } P\boxed{?} - \boxed{?}.$$

There is a penalty for an incorrect answer.

10) (10)

[TLO 3.1, CO 3] We consider a collection of n people who participate in a social network in which pairs of people can be connected, by "friending" each other. The $n \times n$-matrix F is the friend matrix, defined by $F_{ij} = 1$ if persons i and j are friends, and $F_{ij} = 0$ if not. We assume that the friend relationship is symmetric, i.e., person i and person j are friends means person j and person i are friends. We will also assume that $F_{ii} = 0$.

(a) Suppose the n-vector t has components t_i representing the total number of friends of person i. Express t as a matrix-vector product.

(b) Suppose C is the $n \times n$-matrix with C_{ij} equal to the number of friends persons i and j have in common. (Person k is a friend in common of persons i and j if she is a friend of both person i and person j. The diagonal entry C_{ii}, which is the total number of friends person i has in common with herself, is the total number of friends of person i.) Express C as a product of two matrices.

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