Question Paper

Exam Date & Time: 01-Dec-2023 (02:00 PM - 05:00 PM)

MANIPAL ACADEMY OF HIGHER EDUCATION

Manipal School of Information Sciences (MSIS), Manipal
First Semester Master of Engineering - ME (Artificial Intelligence and Machine Learning) Degree Examination - November / December 2023

Applied Linear Algebra [AML 5101]

Marks: 100 **Duration: 180 mins.**

Friday, December 1, 2023

Answer all the questions.

1) (10)

1. [10 points] [CO 1, BT 3] Consider the vectors $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Calculate the following (you may leave the answer in terms of fractions, square root etc.):
   (a) $\text{rms}(x)$.
   (b) $\text{avg}(y)$.
   (c) $\text{std}(x)$.
   (d) The correlation coefficient between x and y.
   (e) The distance between x and y.

2) (10)

2. [10 points] [CO 1, BT 4] The MAHE registrar has the complete list of courses taken by each graduating student in a program. This data is represented as a matrix X with m rows and n columns as follows:

Student \ Course	1	2	$\cdots$	n
1	1	1	$\cdots$	0
2	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

m	1	0	$\cdots$	1

The entries of the data matrix are 1s and 0s representing whether a particular student has taken a particular course. For example, the red-highlighted entry 1 means that the 1st student has taken the 1st course and the blue-highlighted entry 0 means that the mth student has not taken the 2nd course. Recall that the ith student vector is represented as $x^{(i)}$ and the jth course vector is represented as x_j.

(a) The total number of courses the 2nd student has taken is $\boxed{?}^{\mathrm{T}}\boxed{?}$.

(b) In English, explain what the quantity $x_2 \cdot 1$ represents w.r.t. the data?

(c) The total number of students who have taken both class 5 and class 6 is $\boxed{?}^{\mathrm{T}}\boxed{?}$.

(d) In English, explain what the quantity $\left\| x^{(5)} - x^{(6)} \right\|^2$ represents w.r.t. the data?

(e) In English, explain what the quantity $\left[X^{\mathrm{T}} 1\right]_3$ represents w.r.t. the data?

3) (10)

3. [CO 2, BT 4] Suppose that $z_1, z_2, \ldots$ is a time series, with the number z_t giving the value in time t. For example, z_t could be the total sales at a particular store on day t. Consider the following model to predict the future sales z_{t+1} from the previous M sales values:

$$\hat{z}_{t+1} \approx \begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-(M-1)} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix},$$

where $\hat{z}_{t+1}$ denotes the model's prediction of z_{t+1} and the M-vector β contains the model coefficients which have to be computed. For this problem we will assume that $M = 10$. Thus, the model predicts tomorrow's value, given the values over the last 10 days. For each of the following cases of the model coefficients vector β, give a short interpretation in plain English as to what the model predicts as the future sales without referring to mathematical concepts like vectors, dot product, and so on:

(a) $\beta = e_1$;

(b) $\beta = 2e_1 - e_2$;

(c) $\beta = e_6$;

(d) $\beta = 0.5$ … $+ 0.5$ …

(d) $\beta = 0.5e_1 + 0.5e_2$.

4) (10)

4. [CO 2, BT 4] Suppose we represent a 3×3-image as a 9-vector using the order of pixels shown below:

1	4	7
2	5	8
3	6	9

Clearly show the entries of the matrix A such that the matrix-vector product Ax would result in turning the image represented by the vector 9-vector x upside-down?

5) (10)

5. [CO 3, BT 3] Calculate the vector projection of the vector $a = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ on to the direction of the vectors:

   (a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

6) (10)

6. [CO 2, BT 3] Consider the 5×5-matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) For a 5-vector x, how are Ax and x related?

   (b) Compute A^2 defined as the matrix-matrix product $A \times A$. Use the result to identify what A^5 would be without any further calculations.

7) (10)

7. [CO 2, BT 4] A *compartmental system* is a model used to describe the movement of some material over time among a set of n compartments of a system and the outside world. It is widely used in pharmaco-kinetics, the study of how the concentration of a drug varies over time in the body. In this application, the material is a drug, and the compartments are the bloodstream, lungs, heart, liver, kidneys, and so on. Compartmental systems are special cases of linear dynamical systems. In this problem we will consider a very simple compartmental system with 3 compartments. We let $(x_t)_i$ denote the amount of the material (say, a drug) in compartment i at time stamp t. Between time stamps t and $t+1$, the material moves as follows:

- 10% of the material in compartment 1 moves to compartment 2. (This decreases the amount in compartment 1 and increases the amount in compartment 2.)
- 5% of the material in compartment 2 moves to compartment 3.
- 5% of the material in compartment 3 moves to compartment 1.
- 5% of the material in compartment 3 is eliminated.

Express this compartmental system as a matrix-vector product $x_{t+1} = Ax_t$ by clearly showing the entries of the matrix A. Be sure to account for all the material entering and leaving each compartment.

8) (10)

8. [CO 2, BT 3] Consider an n-vector x whose components x_i represent the value of a signal at time stamp $i = 1, 2, \ldots, n$. We want to construct a $2\times$ up-sampled version of the signal x denoted as the $(2n-1)$-vector y using linear interpolation by multiplying x by an appropriate matrix A such that $y = Ax$, where:

$$y_i = \begin{cases} x_{\frac{i+1}{2}}, & \text{if } i \text{ odd}, \\ \frac{1}{2}\left(x_{\frac{i}{2}} + x_{\frac{i}{2}+1}\right), & \text{if } i \text{ even}. \end{cases}$$

Using $n = 5$, write the elements of matrix A.

9) (10)

9. [CO 2, BT 3] Consider the RREF of an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & 8/3 & 1/3 \end{array}\right].$$

(a) Is the underlying system of equations consistent?

(b) Identify the free and pivot variables.

(c) If the system is consistent, express the solution as a set of vectors.

10) (10)

10. [CO 3, BT 4] A train network consists of n stations and m paths, Each station has a delay associated with it which is the time it takes to traverse the station. The delay at each station is not a directly measurable quantity which we denote by the (unknown) n-vector d. However, we can measure the actual total travel time across any route which can be thought of as an approximate measure of the sum of the delays at all stations on that route. For example, if the 1st route contains stations 3, 7, 8, and 9, then the travel time across the 1st route is given by $\underbrace{t_1}_{\text{actual measured travel time}} \approx \underbrace{d_3 + d_7 + d_8 + d_9}_{\text{predicted travel time}}$.

Suppose our goal is to estimate the unknown station delays (that is, the vector d), from a large number of (noisy) measurements of the travel times along all the routes. The associated data is given to you as follows:

- an $m \times n$-matrix P, where

$$P_{ij} = \begin{cases} 1, & \text{if station } j \text{ is on route } i, \\ 0, & \text{otherwise;} \end{cases}$$

- an m-vector t whose entries are the (noisy) measured travel times along the m routes.

You can assume that $m > n$; that is, there are more routes than stations.

(a) What does the jth column of P tell us about the train network?

(b) What does the quantity $P\mathbf{1}$ tell us about the train network?

(c) Write the travel times predicted for all routes as a matrix-vector product.

(d) In order to estimate the unknown station delays d, we want to minimize the deviation between the predicted and the actual travel times. One way to do that is minimize the

$$\underbrace{\text{mean/rms/std/norm/max/min}}_{\text{choose the correct option}} \text{ of } P\boxed{?} - \boxed{?}.$$

There is a penalty for an incorrect answer.

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